NOTES

- NOTES WITH MIND MAPS -

MATHEMATICS

(QUADRATIC EQUATIONS)

10TH

Quadratic Equations

1. Introduction to Quadratic equation

If p(x) is a quadratic polynomial, then p(x) = 0 is called a **quadratic equation**.

The general or standard form of a quadratic equation, in the variable x, is given by $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.

2. Roots of the quadratic equation

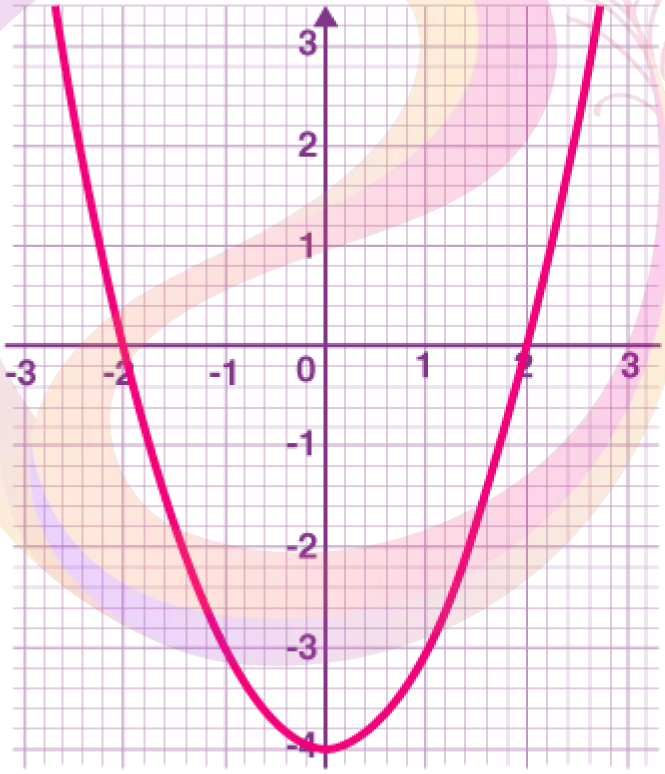
The value of x that satisfies an equation is called the **zeroes** or **roots** of the equation.

A real number α is said to be a solution/root of the quadratic equation $ax^2 + bx + c = 0$ if $a\alpha^2 + b\alpha + c = 0$.

A quadratic equation has **at most two roots**.

Graphically, the roots of a quadratic equation are the points where the graph of the quadratic polynomial cuts the x-axis.

Consider the graph of a quadratic equation $x^2 – 4 = 0$:



Graph of a Quadratic Equation

In the above figure, -2 and 2 are the roots of the quadratic equation x2−4=0

Note:

- If the graph of the quadratic polynomial cuts the x-axis at two distinct points, then it has real and distinct roots.
- If the graph of the quadratic polynomial touches the x-axis, then it has real and equal roots.

- If the graph of the quadratic polynomial does not cut or touch the x-axis then it does not have any real roots.

3. The standard form of a Quadratic Equation

The standard form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$.

'a' is the coefficient of x^2. It is called the quadratic coefficient. 'b' is the coefficient of x. It is called the linear coefficient. 'c' is the constant term.

4. A quadratic equation can be solved by following algebraic methods:

i. Splitting the middle term (factorization)

ii. Completing squares

iii. Quadratic formula

5. Splitting the middle term (or factorization) method

- If $ax^2 + bx + c$, $a \neq 0$, can be reduced to the product of two linear factors, then the roots of the quadratic equation $ax^2 + bx + c = 0$ can be found by equating each factor to zero.
- Steps involved in solving quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ by **splitting the middle term** (or factorization) method:

Step 1: Find the product ac.

Step 2: Find the factors of 'ac' that add to up to b, using the following criteria:

i. If $ac > 0$ and $b > 0$, then both the factors are positive.

ii. If $ac > 0$ and $b < 0$, then both the factors are negative.

iii. If $ac < 0$ and $b > 0$, then larger factor is positive and smaller factor is negative.

iv. If $ac < 0$ and $b < 0$, then larger factor is negative and smaller factor is positive.

Step 3: Split the middle term into two parts using the factors obtained in the above step.

Step 4: Factorize the quadratic equation obtained in the above step by grouping method. Two factors will be obtained.

Step 5: Equate each of the linear factors to zero to get the value of x.

6. Completing the square method

- Any quadratic equation can be converted to the form $(x + a)^2 - b^2 = 0$ or $(x - a)^2 + b^2 = 0$ by adding and subtracting the constant term. This method of finding the roots of quadratic equation is called the method of completing the square.
- The steps involved in solving a quadratic equation by **completing the square**, are as follows:

Step 1: Make the coefficient of x^2 unity.

Step 2: Express the coefficient of x in the form 2 × x × p.

Step 3: Add and subtract the square of p.

Step 4: Use the square identity $(a + b)^2$ or $(a – b)^2$ to obtain the quadratic equation in the required form $(x + a)^2 - b^2 = 0$ or $(x – a)^2 + b^2 = 0$.

Step 5: Take the constant term to the other side of the equation.

Step 6: Take the square root on both the sides of the obtained equation to get the roots of the given quadratic equation.

7. Quadratic formula

The roots of a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ can be calculated by using the **quadratic formula:**

$\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$ where $b^2 – 4ac \geq 0$

If $b^2 – 4ac < 0$, then equation does not have real roots.

The quadratic formula is used to find the roots of a quadratic equation. This formula helps to evaluate the solution of quadratic equations replacing the factorization method. If a quadratic equation does not contain real roots, then the quadratic formula helps to find the imaginary roots of that equation. The quadratic formula is also known as Shreedhara Acharya's formula. In this article, you will learn the quadratic formula, derivation and proof of the quadratic formula, along with a video lesson and solved examples.

An algebraic expression of degree 2 is called the **quadratic equation**. The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b and c are real numbers, also called "numeric coefficients" and $a \neq 0$. Here, x is an unknown variable for which we need to find the solution. We know that the **quadratic formula** used to find the solutions (or roots) of the quadratic equation $ax^2 + bx + c = 0$ is given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here,

a, b, c = Constants (real numbers)

$a \neq 0$

x = Unknown, i.e. variable

The above formula can also be written as:

$$x = \frac{-b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

or

$$x = \frac{-b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

What is the Quadratic Formula used for?

The quadratic formula is used to find the roots of a quadratic equation and these roots are called the solutions of the quadratic equation. However, there are several methods of solving quadratic equations such as factoring, completing the square, graphing, etc.

Roots of Quadratic Equation by Quadratic Formula

We know that a second-degree polynomial will have at most two zeros, and therefore a quadratic equation will have at most two roots.

In general, if α is a root of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$; then, $a\alpha^2 + b\alpha + c = 0$. We can also say that $x = \alpha$ is a solution of the quadratic equation or α satisfies the equation, $ax^2 + bx + c = 0$.

Note: Roots of the quadratic equation ax2 + bx + c = 0 are the same as zeros of the polynomial ax2 + bx + c.

One of the easiest ways to find the roots of a quadratic equation is to apply the quadratic formula.

Quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here, $b^2 - 4ac$ is called the discriminant and is denoted by D.

The sign of plus (+) and minus (-) in the quadratic formula represents that there are two solutions for quadratic equations and are called the roots of the quadratic equation.

Root 1:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

And

Root 2:

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

8. Derivation of Quadratic Formula

We can derive the quadratic formula in different ways using various techniques.

Derivation Using Completing the Square Technique

Let us write the standard form of a quadratic equation.

$ax^2 + bx + c = 0$

Divide the equation by the coefficient of x^2, i.e., a.

$x^2 + (b/a) x + (c/a) = 0$

Subtract c/a from both sides of this equation.

$x^2 + (b/a) x = -c/a$

Now, apply the method of completing the square.

Add a constant to both sides of the equation to make the LHS of the equation as complete square.

Adding $(b/2a)^2$ on both sides,

$x^2 + (b/a) x + (b/2a)^2 = (-c/a) + (b/2a)^2$

Using the identity $a^2 + 2ab + b^2 = (a + b)^2$,

$[x + (b/2a)]^2 = (-c/a) + (b^2/4a^2)$

[x + (b/2a)]2 = (b2 – 4ac)/4a2

Take the square root on both sides,

Shortcut Method of Derivation

Write the standard form of a quadratic equation.

$ax^2 + bx + c = 0$

Multiply both sides of the equation by 4a.

$4a (ax^2 + bx + c) = 4a(0)$

$4a^2x^2 + 4abx + 4ac = 0$

$4a^2x^2 +4abx = -4ac$

Add a constant on sides such that LHS will become a complete square.

Adding b^2 on both sides,

$4a^2x^2 + 4abx + b^2 = b^2 – 4ac$

$(2ax)^2 + 2(2ax)(b) + b^2 = b^2 – 4ac$

Using algebraic identity $a^2 + 2ab + b^2 = (a + b)^2$,

$(2ax + b)^2 = b^2 – 4ac$

Taking square root on both sides,

$2ax + b = \pm \sqrt{(b^2 – 4ac)}$

$2ax = -b \pm \sqrt{(b^2 – 4ac)}$

x = [-b ±√(b2 – 4ac)]/2a

9. Nature of Roots

Based on the value of the discriminant, $D = b^2 - 4ac$, the roots of a quadratic equation can be of three types.

Case 1: If D>0, the equation has two distinct real roots.

Case 2: If D=0, the equation has two equal real roots.

Case 3: If D<0, the equation has no real roots.

The number of roots of a polynomial equation is equal to its degree. So, a quadratic equation has two roots. Some methods for finding the roots are:

Factorization method

Quadratic Formula

Completing the square method

All the quadratic equations with real roots can be factorized. The physical significance of the roots is that at the roots of an equation, the graph of the equation intersects x-axis. The x-axis represents the real line in the Cartesian plane. This means that if the equation has unreal roots, it won't intersect x-axis and hence it cannot be written in factorized form. Let us now go ahead and learn how to determine whether a quadratic equation will have real roots or not.

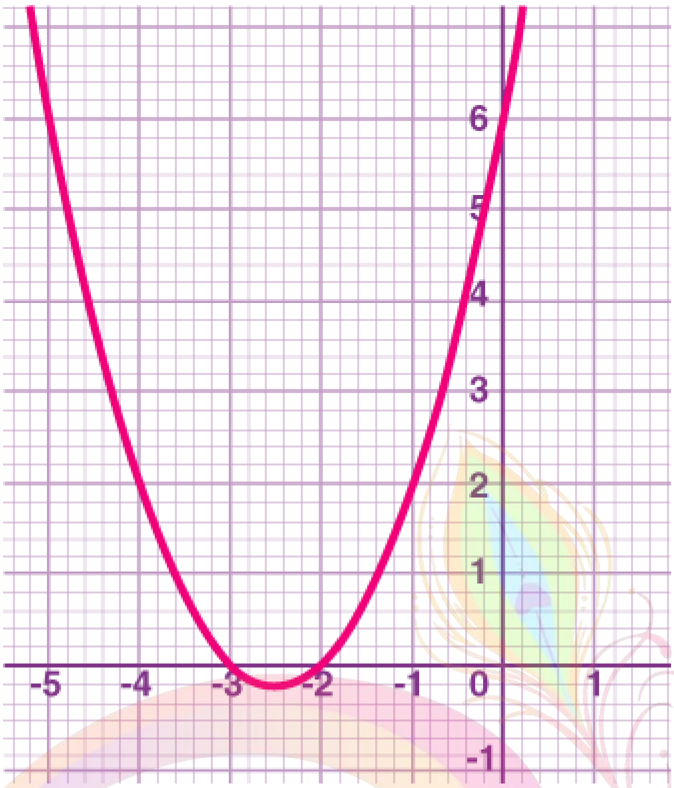
Nature Of Roots Of Quadratic Equation

Value of Discriminant	Nature of Roots	
D > 0	Real, Distinct	
	D is a perfect square	Rational roots
	D is not a perfect square	Irrational roots
D = 0	Real, Equal	
D < 0	Complex, Distinct (A pair of complex conjugates)	

10. Graphical Representation of a Quadratic Equation

The graph of a quadratic polynomial is a parabola. The roots of a quadratic equation are the points where the parabola cuts the x-axis i.e. the points where the value of the quadratic polynomial is zero.

Now, the graph of $x^2 + 5x + 6 = 0$ is:



In the above figure, -2 and -3 are the roots of the quadratic equation

$x^2 + 5x + 6 = 0$.

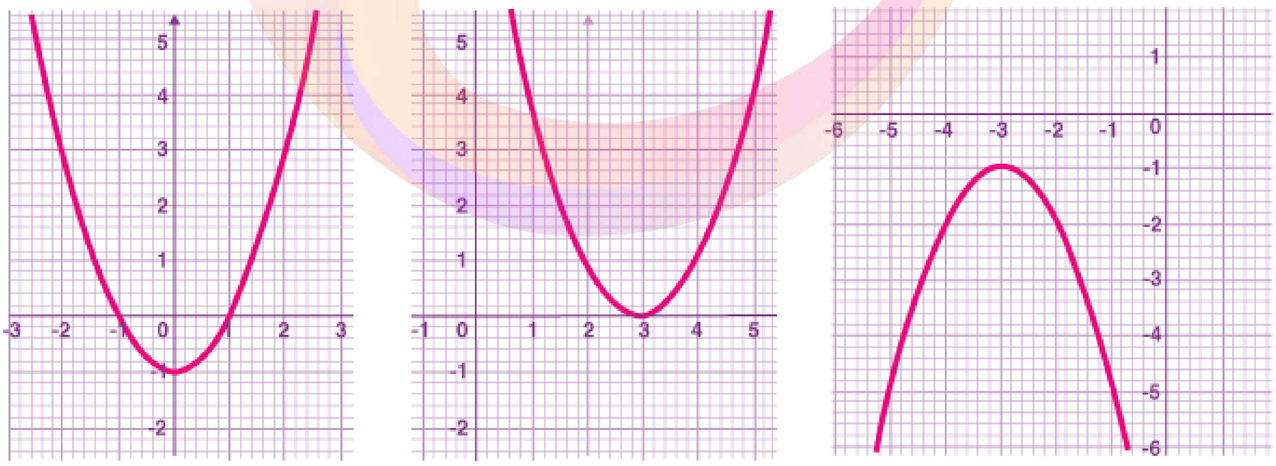
For a quadratic polynomial $ax^2 + bx + c$,

If $a > 0$, the parabola opens upwards.

If $a < 0$, the parabola opens downwards.

If $a = 0$, the polynomial will become a first-degree polynomial and its graph is linear.

The discriminant, $D = b^2 - 4ac$



(a) (b) (c)

Nature of graph for different values of D.

If $D > 0$, the parabola cuts the x-axis at exactly two distinct points. The roots are distinct. This case is shown in the above figure in a, where the quadratic polynomial cuts the x-axis at two distinct points.

If D = 0, the parabola just touches the x-axis at one point and the rest of the parabola lies above or below the x-axis. In this case, the roots are equal.

This case is shown in the above figure in b, where the quadratic polynomial touches the x-axis at only one point.

If D < 0, the parabola lies entirely above or below the x-axis and there is no point of contact with the x-axis. In this case, there are no real roots.

This case is shown in the above figure in c, where the quadratic polynomial neither cuts nor touch the x-axis.

11. Discriminant of a quadratic equation

For the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, the expression $b^2 - 4ac$ is known as **discriminant**.

12. Nature of the roots of a quadratic equation:

- **i.** If $b^2 - 4ac > 0$, the quadratic equation has **two distinct real roots**.
- **ii.** If $b^2 - 4ac = 0$, the quadratic equation has **two equal real roots**.
- **iii.** If $b^2 - 4ac < 0$, the quadratic equation has **no real roots**.

13. There are many equations which are not in the quadratic form but can be reduced to the quadratic form by simplifications.

14. Application of quadratic equations

- The applications of quadratic equation can be utilized in solving real life problems.
- Following points can be helpful in solving word problems:
  - **i.** Every two digit number 'xy' where x is a ten's place and y is a unit's place can be expressed as $xy = 10x + y$.
  - **ii.** Downstream: It means that the boat is running in the direction of the stream Upstream: It means that the boat is running in the opposite direction of the stream Thus, if Speed of boat in still water is x km/h

    And the speed of stream is y km/h

    Then the speed of boat downstream will be (x + y) km/h and in upstream it will be (x − y) km/h.
  - **iii.** If a person takes x days to finish a work, then his one day's work $= \frac{1}{x}$.