NOTES

– NOTES WITH MIND MAPS –

MATHEMATICS

(INTEGRALS)

12TH

INTEGRALS

Top Concepts

1. Integration is the inverse process of differentiation. The process of finding the function from its primitive is known as integration or antidifferentiation.

2. The problem of finding a function whenever its derivative is given leads to indefinite form of integrals.

3. The problem of finding the area bounded by the graph of a function under certain conditions leads to a definite form of integrals.

4. Indefinite and definite integrals together constitute **Integral Calculus**.

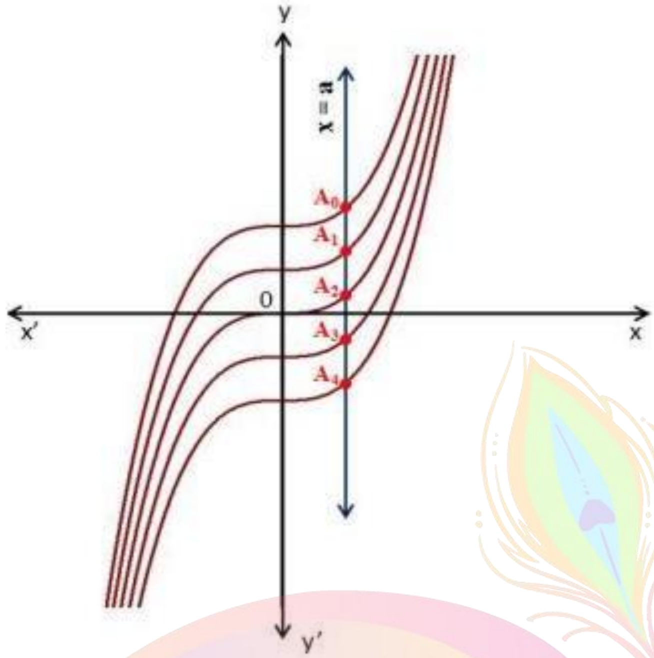
5. Indefinite integral $\int f(x)dx = F(x) + C$, where F(x) is the antiderivative of f(x).

6. Functions with same derivatives differ by a constant.

7. $\int f(x)dx$means integral of f with respect to x, f(x) is the integrand, x is the variable of integration and C is the constant of integration.

8. Geometrically indefinite integral is the collection of family of curves, each of which can be obtained by translating one of the curves parallel to itself.

   Family of curves representing the integral of $3x^2$



$\int f(x)dx = F(x) + C$, represents a family of curves where different values of C correspond to different members of the family, and these members are obtained by shifting any one of the curves parallel to itself.

9. **Properties of antiderivatives**

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$\int kf(x)dx = k\int f(x)dx$ for any real number k

$$\int [k_1f_1(x) + k_2f_2(x) + \ldots\ldots + k_nf_n(x)]dx = k_1\int f_1(x)dx + k_2\int f_2(x)dx + \ldots. + k_n\int f_n(x)dx$$

where $k_1, k_2, \ldots k_n$ are real numbers and $f_1, f_2, \ldots f_n$ are real functions.

10. Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.

11. **Comparison between differentiation and integration**

  1. Both are operations on functions.
  2. Both satisfy the property of linearity.
  3. All functions are not differentiable and all functions are not integrable.
  4. The derivative of a function is a unique function, but the integral of a function is not.
  5. When a polynomial function P is differentiated, the result is a polynomial whose degree is 1 less than the degree of P. When a polynomial function P is integrated, the result is a polynomial whose degree is 1 more than that of P.

6. The derivative is defined at a point P and the integral of a function is defined over an interval.
7. Geometrical meaning: The derivative of a function represents the slope of the tangent to the corresponding curve at a point. The indefinite integral of a function represents a family of curves placed parallel to each other having parallel tangents at the points of intersection of the family with the lines perpendicular to the axis.
8. The derivative is used for finding some physical quantities such as the velocity of a moving particle when the distance traversed at any time t is known. Similarly, the integral is used in calculating the distance traversed when the velocity at time I is known.
9. Differentiation and integration, both are processes involving limits.
10. By knowing one antiderivative of function f, an infinite number of antiderivatives can be obtained.

12. Integration can be done by using many methods. Prominent among them are

 1. Integration by substitution
 2. Integration using partial fractions
 3. Integration by parts
 4. Integration using trigonometric identities.

13. A change in the variable of integration often reduces an integral to one of the fundamental integrals. Some standard substitutions are

$x^2 + a^2$; substitute $x = a \tan \theta$

$\sqrt{x^2 - a^2}$; substitute $x = a \sec \theta$

$\sqrt{a^2 - x^2}$; substitute $x = a \sin \theta$ or $a \cos \theta$

14. A function of the form $\frac{P(x)}{Q(x)}$ is known as a rational function. Rational functions can be integrated using partial fractions.

15. **Partial fraction decomposition or partial fraction expansion** is used to reduce the degree of either the numerator or the denominator of a rational function.

16. **Integration using partial fractions**

A rational function $\frac{P(x)}{Q(x)}$can be expressed as the sum of partial fractions if $\frac{P(x)}{Q(x)}$. This takes any of the forms:

- $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$, $a \neq b$
- $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$
- $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
- $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
- $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

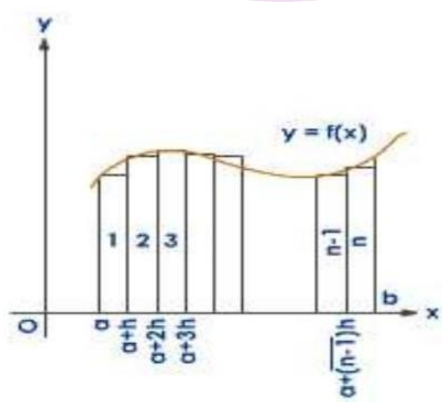
where x^2+bx+c cannot be factorised further.

17. To find the integral of the product of two functions, integration by parts is used. I and II functions are chosen using the ILATE rule:
    I - inverse trigonometric
    L - logarithmic
    A - algebraic
    T - trigonometric
    E - exponential is used to identify the first function.

18. **Integration by parts**
    Integral of the product of two functions = (first function) × (integral of the second function) - integral of [(differential coefficient of the first function) × (integral of the second function)].

$$\int f_1(x).f_2(x)dx = f_1(x)\int f_2(x)dx - \int\left[\frac{d}{dx}f_1(x).\int f_2(x)dx\right]dx$$, where f_1 and f_2 are functions of x.

19. Definite integral $\int_a^b f(x)dx$ of the function f(x) from limits a to b represents the area enclosed by the graph of the function f(x), the x-axis and the vertical markers x = 'a' and x = 'b'.



20. **Definite integral as the limit of a sum:** The process of evaluating a definite integral by using the definition is called integration as the limit of a sum or integration from first principles.

21. Method of evaluating $\int_a^b f(x)dx$

   i. Calculate antiderivative F(x)

   ii. Calculate F(b) - F(a)

22. Area function

$$A(x) = \int_a^x f(x)dx \text{, if x is a point in [a, b].}$$

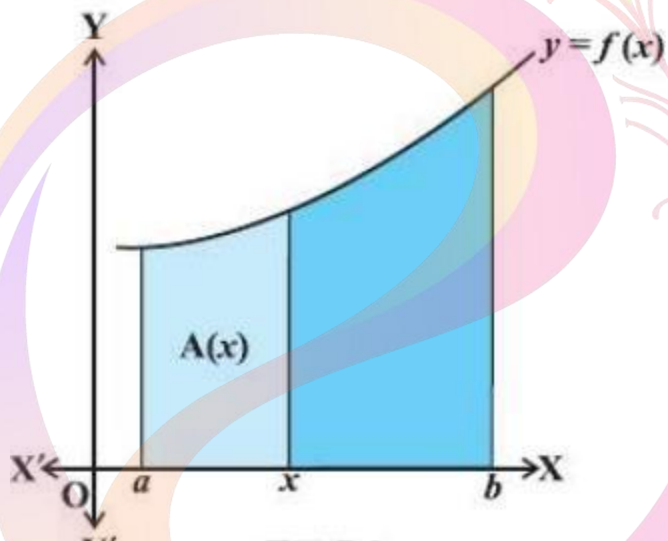


Fig 7.3

23. Fundamental Theorem of Integral Calculus

- **First fundamental theorem** of integral calculus: If area function, $A(x) = \int_a^x f(x)dx$ for all x ≥ a, and f is continuous on [a, b]. Then A'(x) = f(x) for all x ∈ [a, b]

- **Second fundamental theorem** of integral calculus: Let f be a continuous function of x in the closed interval [a, b] and let F be antiderivative of $\frac{d}{dx}f(x) = f(x)$ for all x in domain of f, then

$$\int_a^b f(x)dx = [F(x) + C]_a^b = F(b) - F(a)$$

Top Formulae

1. **Some Standard Integrals**

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
- $\int dx = x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \sin x \, dx = -\cos x + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$
- $\int \sec x \tan x \, dx = \sec x + C$
- $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$
- $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
- $\int -\frac{dx}{\sqrt{1-x^2}} = \cos^{-1} x + C$
- $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
- $\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$
- $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$
- $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + C$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\log a} + C$

- $\int \frac{1}{x} dx = \log|x| + C$
- $\int \tan x \, dx = \log|\sec x| + C$
- $\int \cot x \, dx = \log|\sin x| + C$
- $\int \sec x \, dx = \log|\sec x + \tan x| + C$
- $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + C$
- $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$
- $\int \frac{1}{ax+b} dx = \frac{1}{a} \log|ax+b| + C$
- $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$
- $\int a^{bx+c} dx = \frac{1}{b} \cdot \frac{a^{bx+c}}{\log a} + C, a > 0, a \neq 1$
- $\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b) + C$
- $\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + C$
- $\int \tan(ax+b) dx = \frac{1}{a} \log|\sec(ax+b)| + C$
- $\int \cot(ax+b) dx = \frac{1}{a} \log|\sin(ax+b)| + C$
- $\int \sec^2(ax+b) dx = \frac{1}{a} \tan(ax+b) + C$
- $\int \operatorname{cosec}^2(ax+b) dx = -\frac{1}{a} \cot(ax+b) + C$
- $\int \sec(ax+b) \tan(ax+b) dx = \frac{1}{a} \sec(ax+b) + C$
- $\int \operatorname{cosec}(ax+b) \cot(ax+b) dx = -\frac{1}{a} \operatorname{cosec}(ax+b) + C$
- $\int \sec(ax+b) dx = \frac{1}{a} \log|\sec(ax+b) + \tan(ax+b)| + C$
- $\int \operatorname{cosec}(ax+b) dx = \frac{1}{a} \log|\operatorname{cosec}(ax+b) - \cot(ax+b)| + C$