NOTES

- NOTES WITH MIND MAPS -

MATHEMATICS

(CONTINUITY AND DIFFERENTIABILITY)

12TH

CONTINUITY & DIFFERENTIABILITY

Top Definitions

1. A function f(x) is said to be continuous at a point c if.

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

2. A real function f is said to be continuous if it is continuous at every point in the domain of f.

3. If f and g are real-valued functions such that (f o g) is defined at c, then (f g)(x) - f(g(x)).
   If g is continuous at c and if f is continuous at g(c), then (f o g) is continuous at c.

4. A function f is differentiable at a point c if Left Hand Derivative (LHD) = Right Hand Derivative (RHD),

i.e. $\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h}$

5. If a function f is differentiable at every point in its domain, then

$\lim_{h \to 0} \frac{f(x+h)-f(c)}{h}$ or $\lim_{h \to 0} \frac{f(x-h)-f(c)}{-h}$ is called the derivative or differentiation of f at x and is denoted by $f'(x)$ or $\frac{d}{dx} f(x)$.

6. If LHD ≠ RHD, then the function f(x) is not differentiable at x = c.

7. Geometrical meaning of differentiability:
   The function f(x) is differentiable at a point P if there exists a unique tangent at point P. In other words, f(x) is differentiable at a point P if the curve does not have P as its corner point.

8. A function is said to be differentiable in an interval (a, b) if it is differentiable at every point of (a, b).

9. A function is said to be differentiable in an interval [a, b] if it is differentiable at every point

of [a, b].

10. **Chain Rule of Differentiation:** If f is a composite function of two functions u and v such that f = v(t) and

t = u(x) and if both $\frac{dv}{dt}$ and $\frac{dt}{dx}$ exist, then $\frac{dv}{dx} = \frac{dv}{dt}.\frac{dt}{dx}$.

11. Logarithm of a to the base b is x, i.e., $\log_b a = x$ if $b^x = a$, where b > 1 is a real number. Logarithm of a to base b is denoted by $\log_b a$.

12. Functions of the form x = f(t) and y = g(t) are parametric functions.

13. **Rolle's Theorem:** If f : [a, b] → R is continuous on [a, b] and differentiable on (a, b) such that f(a) = f(b), then there exists some c in (a, b) such that f'(c) = 0.

14. **Mean Value Theorem:** If f : [a, b] → R is continuous on [a, b] and differentiable on (a, b), then thereexists some c in (a, b) such that $f'(c) = \lim_{h\to 0} \frac{f(b)-f(a)}{b-a}$.

Top Concepts

1. A function is continuous at x = c if the function is defined at x = c and the value of the function at x = c equals the limit of the function at x = c.
2. If function f is not continuous at c, then f is discontinuous at c and c is called the point of discontinuity of f.
3. Every polynomial function is continuous.
4. The greatest integer function [x] is not continuous at the integral values of x.
5. Every rational function is continuous.

   Algebra of continuous functions:

   i. Let f and g be two real functions continuous at a real number c, then f + g is continuous at x = c.

   ii. f - g is continuous at x = c.

   iii. f . g is continuous at x = c.

   iv. $(\frac{f}{g})$ is continuous at x = c, [provided g(c) ≠ 0].

v. kf is continuous at x = c, where k is a constant.

6. Consider the following functions:
   i. Constant function
   ii. Identity function
   iii. Polynomial function
   iv. Modulus function
   v. Exponential function
   vi. Sine and cosine functions

   The above functions are continuous everywhere.

7. Consider the following functions:
   i. Logarithmic function
   ii. Rational function
   iii. Tangent, cotangent, secant and cosecant functions

   The above functions are continuous in their domains.

8. If f is a continuous function, then |f| and $\frac{1}{f}$ are continuous in their domains.

9. Inverse functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\operatorname{cosec}^{-1}x$ and $\sec^{-1}x$ are continuous functions on their respective domains.

10. The derivative of a function f with respect to x is f'(x) which is given by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

11. If a function f is differentiable at a point c, then it is also continuous at that point.

12. Every differentiable function is continuous, but the converse is not true.

13. Every polynomial function is differentiable at each $x \in R$.

14. Every constant function is differentiable at each $x \in R$.

15. The chain rule is used to differentiate composites of functions.

16. The derivative of an even function is an odd function and that of an odd function is an even function.

17. **Algebra of Derivatives**

If u and v are two functions which are differentiable, then

i. $(u \pm v)' = u' \pm v'$ (Sum and Difference Formula)

ii. $(uv)' = u'v + uv'$ (Leibnitz rule or Product rule)

iii. $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, $v \neq 0$,(Quotient rule)

18. **Implicit Functions**

If it is not possible to separate the variables x and y, then the function f is known as an implicit function.

19. **Exponential function:** A function of the form $y = f(x) = b^x$, where base $b > 1$.
   1. Domain of the exponential function is R, the set of all real numbers.
   2. The point (0, 1) is always on the graph of the exponential function.
   3. The exponential function is ever increasing.

20. The exponential function is differentiable at each $x \in R$.

21. Properties of logarithmic functions:
   i. Domain of log function is R^+.
   ii. The log function is ever increasing.
   iii. For 'x' very near to zero, the value of log x can be made lesser than any given real number.

22. Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = [u(x)]^{v(x)}$. Here both f(x) and u(x) need to be positive.

23. To find the derivative of a product of a number of functions or a quotient of a number of functions, take the logarithm of both sides first and then differentiate.

24. **Logarithmic Differentiation**

$y = a^x$

Taking logarithm on both sides

$\log y = \log a^x$.

Using the property of logarithms

$\log y = x \log a$

Now differentiating the implicit function

$$\frac{1}{y}.\frac{dy}{dx} = \log a$$

$$\frac{dy}{dx} = y\log a = a^x \log a$$

25. The logarithmic function is differentiable at each point in its domain.

26. Trigonometric and inverse-trigonometric functions are differentiable in their respective domains.

27. The sum, difference, product and quotient of two differentiable functions are differentiable.

28. The composition of a differentiable function is a differentiable function.

29. A relation between variables x and y expressed in the form x = f(t) and y = g(t) is the parametric form with t as the parameter. Parametric equation of parabola $y^2 = 4ax$ is $x = at^2$, y = 2at.

30. Differentiation of an infinite series: If f(x) is a function of an infinite series, then to differentiate the function f(x), use the fact that an infinite series remains unaltered even after the deletion of a term.

31. **Parametric Differentiation:**

Differentiation of the functions of the form x = f(t) and y = g(t):

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

32. Let u = f(x) and v = g(x) be two functions of x. Hence, to find the derivative of f(x) with respect g(x), we use the following formula:

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

33. If y = f(x) and $\frac{dy}{dx} = f'(x)$ and if f'(x) is differentiable, then

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ or f''(x) is the second order derivative of y with respect to x.

34. If x = f(t) and y = g(t), then

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{g'(t)}{f'(t)}\right\}$$

$$\text{or } \frac{d^2y}{dx^2} = \frac{d}{dt}\left\{\frac{g'(t)}{f'(t)}\right\} \cdot \frac{dt}{dx}$$

$$\text{or, } \frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{\{f'(t)\}^3}$$

Top Formulae

1. Derivative of a function at a point

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. Properties of logarithms

$$\log(xy) = \log x + \log y$$

$$\log\left(\frac{x}{y}\right) = \log x - \log y$$

$$\log(x^y) = y \log x$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

3. Derivatives of Functions

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a}, a > 0, a \neq 1$$

$$\frac{d}{dx}\left(a^x\right) = a^x \log_e a, a > 0$$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}, \text{ if } |x| > 1$$

$$\frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}}, \text{ if } |x| > 1$$

$$\frac{d}{dx}\left\{\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right\} = \begin{cases} -\frac{2}{1+x^2}, x > 1 \\ \frac{2}{1+x^2}, -1 < x < 1 \\ -\frac{2}{1+x^2}, x < -1 \end{cases}$$

$$\frac{d}{dx}\left\{\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right\} = \begin{cases} \frac{2}{1+x^2}, x > 0 \\ \frac{-2}{1+x^2}, x < 0 \end{cases}$$

$$\frac{d}{dx}\left\{\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\} = \begin{cases} \frac{2}{1+x^2}, x < -1 \text{ or } x > 1 \\ \frac{2}{1+x^2}, -1 < x < 1 \end{cases}$$

$$\frac{d}{dx}\left\{\sin^{-1}\left(3x-4x^3\right)\right\}=\begin{cases}-\frac{3}{\sqrt{1-x^2}}, \frac{1}{2}<x<1, -1<x<-\frac{1}{2}\\ \frac{3}{\sqrt{1-x^2}}, -\frac{1}{2}<x<\frac{1}{2}\end{cases}$$

$$\frac{d}{dx}\left\{\cos^{-1}\left(4x^3-3x\right)\right\}=\begin{cases}-\frac{3}{\sqrt{1-x^2}}, \frac{1}{2}<x<1\\ \frac{3}{\sqrt{1-x^2}}, -\frac{1}{2}<x<\frac{1}{2} \text{ or } -1<x<-\frac{1}{2}\end{cases}$$

$$\frac{d}{dx}\left\{\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)\right\}=\begin{cases}\frac{3}{1+x^2}, x<-\frac{1}{\sqrt{3}} \text{ or } x>\frac{1}{\sqrt{3}}\\ \frac{3}{1+x^2}, -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}\end{cases}$$

$$\frac{d}{dx}\left[\sin\left(\sin^{-1}x\right)\right]=1, \text{ if } -1<x<1$$

$$\frac{d}{dx}\left[\cos\left(\cos^{-1}x\right)\right]=1, \text{ if } -1<x<1$$

$$\frac{d}{dx}\left[\tan\left(\tan^{-1}x\right)\right]=1, \text{ for all } x\in R$$

$$\frac{d}{dx}\left[\operatorname{cosec}\left(\operatorname{cosec}^{-1}x\right)\right]=1, \text{ for all } x\in R-(-1,1)$$

$$\frac{d}{dx}\left[\sec\left(\sec^{-1}x\right)\right]=1, \text{ for all } x\in R-(-1,1)$$

$$\frac{d}{dx}\left[\cot\left(\cot^{-1}x\right)\right]=1, \text{ for all } x\in R$$

$$\frac{d}{dx}\left[\sin^{-1}\left(\sin x\right)\right]=\begin{cases}-1, -\frac{3\pi}{2}<x<-\frac{\pi}{2}\\ 1, -\frac{\pi}{2}<x<\frac{\pi}{2}\\ -1, \frac{\pi}{2}<x<\frac{3\pi}{2}\\ 1, \frac{3\pi}{2}<x<\frac{5\pi}{2}\end{cases}$$